§6.1

23. $V=\mathbb{F}^n$, $A\in M_{n\times n}(\mathbb{F})$ $(\mathbb{F}=\mathbb{R}$ or $\mathbb{C})$

a) Prove that $\langle x, Ay\rangle = \langle A^*x, y\rangle$, for all $x, y\in V$

b) Suppose that for some $B\in M_{n\times n}(\mathbb{F})$, we have $\langle x, Ay\rangle = \langle Bx, y\rangle$ for all $x, y\in V$. Prove that $B=A^*$

c) Let α be the standard ordered basis for V. For any orthonormal basis β for V. Let Q be the $n\times n$ matrix whose columns are the vectors in β. Prove that $Q^*=Q^{-1}$

d) Define linear operators T and U on V by $T(x)=Ax$ and $U(x)=A^*x$. Show that $[U]_\beta=[T]_\beta^*$ for any orthonormal basis β for V.

Solution: a) With standard inner product, $\langle x, y\rangle = y^*x$.

$$\langle x, Ay\rangle = (Ay)^*x = y^*A^*x = \langle A^*x, y\rangle$$

b) $\langle A^*x, y\rangle = \langle x, Ay\rangle = \langle Bx, y\rangle$ for all $x, y\in V$.

Thm 6.1(e) $\Rightarrow A^*x = Bx$ for all x. $\Rightarrow A^*=B$

c) $\beta=\{v_1, \dots v_n\}$.

$$(Q^*Q)_{ij} = v_i^* v_j = \langle v_i, v_j\rangle = \begin{cases} 1 & i=j \\ 0 & i\neq j\end{cases} \Rightarrow Q^*Q=I \Rightarrow Q^*=Q^{-1}$$

d) α — standard basis for $\mathbb{F}^n$. $[T]_\alpha=A$ $[U]_\alpha=A^*$

$[I]_\alpha^\beta = Q$ (in c))

$$[U]_\beta = [I]_\alpha^\beta[U]_\alpha[I]_\beta^\alpha = QAQ^{-1} = QAQ^* = [I]_\alpha^\beta[T]_\alpha[I]_\beta^\alpha = [T]_\beta$$

§6.2

22. $V=C([0,1])$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$. Let W be the subspace spanned by the linearly independent set $\{t, \sqrt{t}\}$

a) Find an orthonormal basis for W.

b) $h(t):=t^2$. Use the basis in a) to obtain the closest approximation of h in W.

Solution: a) $v_1'=t$. $\quad v_2' = \sqrt{t} - \dfrac{\langle\sqrt{t}, t\rangle}{\langle t, t\rangle}t = -\dfrac{6t-5\sqrt{t}}{5}$

Normalize $\Rightarrow$ $\{\underbrace{\sqrt{3}t}_{v_1}, \underbrace{-\sqrt{2}(6t-5\sqrt{t})}_{v_2}\}$

b) $\langle h, v_1\rangle = \frac{3}{4}t$ $\qquad \langle h, v_2\rangle = \frac{6}{7}t - \frac{5}{7}\sqrt{t}$

The app'n is $-\dfrac{20\sqrt{t}-45t}{28}$

§6.3

19. Suppose that A is an $n \times n$ matrix in which no two columns are identical. Prove that A^*A is a diagonal matrix iff every pair of columns of A are orthogonal

Solution: v_i — i-th column of A $\quad v_i^*$ ~ i-th row of A^*

$$(A^*A)_{ij} = v_i^* v_j = \langle v_j, v_i \rangle = \delta_{ij}$$

$\Rightarrow$ Conclusion

§6.4

14. Let V be a fin-dim real inner product space, U, T self-adjoint lin. op. on V s.t $TU = UT$. Prove that there exists an orthonormal basis for V consisting of vectors that are eigenvectors of both U and T.

Solution: Induction on $\dim V$ $\quad n := \dim V$

- $n = 1$, U and T will be diagonalized by any orthnm. basis simultaneously.
- Suppose the statement true for $n \leq k-1$, prove it for $n = k$.
- Pick an arbitrary eigenspace $W = E_\lambda$ of T for some eigenvalue λ.
- W is both U-invariant and T-invariant. (why?)
- If $W = V$, then Thm 6.17 to $U \Rightarrow$ an orthnm. basis β consisting of eigenvectors of U which are also eigenvectors of V.
- If W is a proper subspace of V. then by induction hypothesis applying to T_W, U_W (Note that T_W and U_W are self-adjoint, cf. Ex 6.4.7) we have an orthnm. basis β_1 for W consisting of eigenvectors of T and U.
- $W^\perp$ is also T-inv. and U-inv. (cf. Ex 6.4.7)
  By induction hypothesis, $\Rightarrow$ an orthnm. basis for $W^\perp$ consisting of eigenvectors of T and U.
- V is fin dim. $\beta = \beta_1 \cup \beta_2$ is an orthnm basis for V consisting of eigenvectors of T and U.

5. Let A and B be symmetric $n \times n$ matrixes s.t. $AB = BA$. Use Ex 14 to prove that there exists an orthnm ~~basis~~ matrix P s.t P^tAP and P^tBP are both diagonal

Solution: $T := L_A \quad U := L_B$

Ex 14 $\Rightarrow$ $\exists$ orthnm basis β s.t. $[T]_\beta$ and $[U]_\beta$ are diagonal.

α - standard basis.

$[T]_\beta = [I]_\alpha^\beta A [I]_\beta^\alpha \qquad [U]_\beta = [I]_\alpha^\beta B [I]_\beta^\alpha$ are diagonal

$P := [I]_\beta^\alpha$

§6.5

13. Suppose that A and B are diagonlizable matrices. Prove or disprove that A is similar to B iff A and B are unitarily equivalent.

Solution $\Rightarrow$ is false.

$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are similar.

They are not unitarily equ. since one is symmetric and the other is not.

§6.6

2. $V = \mathbb{R}^2$. $W = \mathrm{Span}(\{(1,2)\})$. β the standard ordered basis for V. Compute $[T]_\beta$ where T is the orthogonal projection of V and W. Do the same for $V = \mathbb{R}^3$ and $W = \mathrm{Span}\{(1,0,1)\}$

Solution: • The projection of $(1,0)$:

$$\frac{\langle (1,0), (1,2) \rangle}{\|(1,2)\|^2}(1,2) = \frac{1}{5}(1,2)$$

That of $(0,1)$:

$$\frac{\langle (0,1), (1,2) \rangle}{\|(1,2)\|^2}(1,2) = \frac{2}{5}(1,2)$$

$$\Rightarrow [T]_\beta = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

• $$\frac{\langle (1,0,0), (1,0,1) \rangle}{\|(1,0,1)\|^2}(1,0,1) = \frac{1}{2}(1,0,1)$$

$$\frac{\langle (0,1,0), (1,0,1) \rangle}{\|(1,0,1)\|^2}(1,0,1) = 0(1,0,1)$$

$$\frac{\langle (0,0,1), (1,0,1) \rangle}{\|(1,0,1)\|^2}(1,0,1) = \frac{1}{2}(1,0,1)$$

$$\Rightarrow [T]_\beta = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$